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Solving Ordinary Differential Equation Using Least Square Method and Conjugate Gradient Method

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ABSTRACT

In this study, the nonhomogeneous Ordinary differential equation (ODE) with boundary value problem is addressed (BVP). It commonly appeared in a variety of fields and professions such as engineering and physics. The objectives of this project are two. First is to find the exact solution using Undetermined coefficient (UC) and Variation of Parameters (VP). Second is to find the best approximation of the second order linear ode using the Least square method (LSM) and Conjugate method (CG). The theoretical method used to solve this problem is the undetermined coefficient which is really complicated to understand and will take a longer time to solve the problems. This study proceeds to solve the problems using two numerical methods which are the Least Square Method (LSM) and the Conjugate Gradient (CG). CG is used to solve the inverse matrix to avoid the ill-conditioned matrix. Numerical solution shows that the Least Square Method can be used to solve the second-order nonhomogeneous ordinary differential equation with BVP based on the term of the error analysis made by the theoretical method which is the undetermined coefficient. This study demonstrates that when the answers are displayed on the same graph, the theoretical technique and numerical methods resemble each other. Second-order linear ODE has a variety of applications to model problems in science and engineering. The demand for the applications of a simpler technique is widespread in science, especially with rapidly increasing fields in physics, chemistry, and biology.

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1. Introduction

A differential equation is an equation which contains derivative, either ordinary derivatives or partial derivatives and essential tools in a wide range of application [1]. Other than that differential equation is often classified with respect to order. The order of a differential equation is the order of the highest order derivative present in equation.

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$$m\frac{dv}{dt} = F(t, v) \quad (1)$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \quad (2)$$

For example, the equation (1) is a first order differential equation, and the equation (2) is the second order differential equation. Another equation of second order differential equation formulated as in Equation (3) [2].

$$ay'' + by' + cy = g(t) \quad (3)$$

There are two classes of the differential equation which are Linear Differential Equation and Non-Linear Differential Equation [3]. The general form of the second order linear ordinary differential equation for the function y is given in Equation (4).

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \quad (4)$$

where $a_2, a_1, a_0$are constants. If f(x) = 0, this equation is said be a second order homogenous ODE and if f(x) ≠ 0, this equation is said to be a second order nonhomogeneous ODE. Theoretically, the homogeneous part of the equation is easier to solve as it is calculated based on the quadratic formula as in Equation (5) [4].

$$m = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a} \quad (5)$$

.

However, for the nonhomogeneous part, it requires an understanding of calculus and consumes time to solve the solution. There are also have a complicated function such as logarithm, exponential, and trigonometry to calculate. There are two common theoretical methods in solving second order nonhomogeneous linear ODE are known as Undetermined Coefficient and Variation of Parameter [5]. This method is applicable to differential equation with variable coefficients and for functions other than exponential, trigonometric and algebraic. This method does include integration which is sometimes too difficult to solve. There are two approaches to estimate constant or parameters in ODE which are Initial-Value Problem (IVP) and Boundary Value Problem (BVP) [6]. IVP which is when the conditions of a solution must satisfy is specified at one value of the independent variable. In the field of differential equation, IVP together with specified value called initial condition [7]. Meanwhile, boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions [8].

2. Literature Review

There are several methods in solving second order nonhomogeneous ODE but sometimes it is too difficult to use as it carried out too many complicated steps and may not necessarily be directly solvable. There are two common theoretical methods in solving second order nonhomogeneous linear ODE are known as Undetermined Coefficient and Variation of Parameter [9]. This method is applicable to differential equation with variable coefficients and functions other than exponential, trigonometric and algebraic but this method does include integration which is sometimes too difficult to solve. There are two approaches to estimate constant or parameters in ODE which are Initial-Value Problem (IVP) and Boundary Value Problem (BVP). IVP which is when the conditions of a solution must satisfy is specified at one value of the independent variable. In the field of differential equation, IVP together with specified value called initial condition.

Numerical methods can be applied to approximate solutions for second order linear nonhomogeneous ODE [10]. Since many problems can be modeled by a relationship between a function and its derivatives. With the approach of Least Square Method, the problem can be easily solved as this method is practical, fast and easy to implement [11]. Meaning of the Least Square is the sum of squares has to be minimized. The best approximate solution is determined by finding the minimum value of error when compared to the exact solution which is using theoretical method. The ends process of the least square method results in inverse matrix in non-singular form. The matrix is not invertible when the matrix is singular or nearly singular which if the determinant of a matrix is zero or approaching zero. To solve this problem, the optimization method of the Conjugate Gradient will be applied [12].

3. Methodology

The methodology of this research started by identifying the ODE function. The flow chart in Figure 1 presents the steps. Furthermore, Table 1 lists the nonhomogeneous ODE problem/functions with BVP that has been chosen as the guideline in this research.

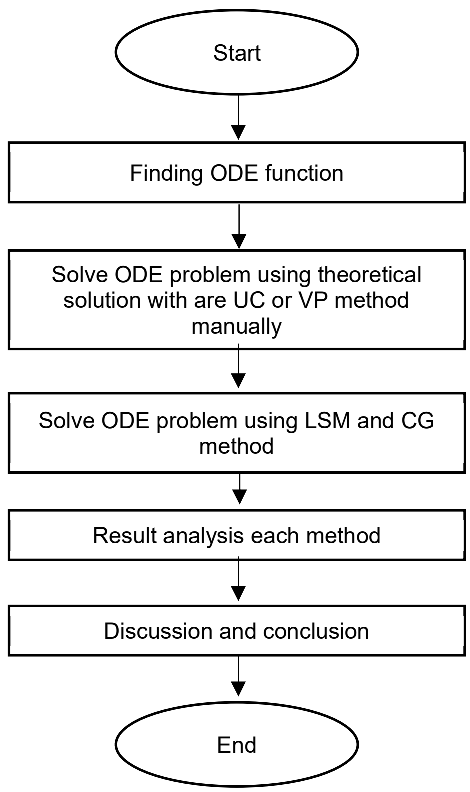

Figure 1. Research Flow Chart

Table 1. List of OBE Functions

No.	Function	BVP	Method Selected
1.	$y'' - 3y' - 4y = 3e^{2x}$	$y(0) = 1$ $y(1) = 3$	Undetermined coefficient (UC)
2.	$y'' - 3y' - 4y = 2\sin x$	$y(0) = 2$ $y(1) = 2$	Undetermined coefficient (UC)
3.	$y'' - 5y' + 4y = e^{3x}$	$y(0) = 1$ $y(1) = 2$	Variation of Parameter (VP)

To see the behaviour of the method, different problems of second order ODE have been observed in this first step then the nonhomogeneous ODE problem with BVP has been chosen. Then, the ODE problem was solved by using theoretical solutions [13], including Undetermined Coefficient (UC) or Variation of parameters (VP) as the second step. Then, solving the ODE functions by using numerical method has been conducted in the third step. There are two numerical methods have been used in this research namely Least Square Method (LSM) and Conjugate gradient (CG). The solution from LSM is an approximate value and must be compared to exact value retrieve from theoretical method. LSM will lead to solving system of linear equation. The possibilities of getting singular or non-singular matrix are high because this system involve of matrix. To avoid this problem, the CG method was applied [14].

4. Results and Discussion

This part provides the theoretical and numerical solutions for every function listed in Table 2, Table 3 and Table 4. Then, graph charts are given to present the comparison between the theoretical method and numerical method after the substitution of the boundary condition into the final solution.

Table 2. Solutions for function $y'' - 3y' - 4y = 3e^{2x}$

Method	Solution
Theoretical method	$y = 0.1132708508e^{4x} + 1.386729149e^{-x} - \frac{1}{2}e^{2x}$
Least Square Method	$y = 1 - 0.95499489x - 6.26384018x^2 + 9.21883508x^3$
Conjugate gradient	$y = 1 - 0.95499403x - 6.26384281x^2 + 9.21883684x^3$

Table 3. Solutions for function $y'' - 3y' - 4y = 2\sin x$

Method	Solution
Theoretical method	$y = 0.02731509926e^{4x} + 1.796214313e^{-x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x$
Least Square Method	$y = 2 - 1.71931843x - 0.85336411x^2 + 2.57268254x^3$
Conjugate gradient	$y = 2 - 1.71931934x - 0.85336412x^2 + 2.57268076x^3$

Table 4. Solutions for function $y'' - 5y' + 4y = e^{3x}$

Method	Solution
Theoretical method	$y = 0.1535344247e^{4x} + 1.346465575e^{x} - \frac{1}{2}e^{3x}$
Least Square Method	$y = 1 + 1.30490832 - 2.11461753x^2 + 1.80970921x^3$
Conjugate gradient	$y = 1 + 1.30490815 - 2.11464630x^2 + 1.80970815x^3$

The comparison between the theoretical method and numerical method are shown in Figure 2, Figure 3 and Figure 4. The results of using the undetermined coefficient variation are quite similar with the numerical methods as shown in Figure 2 and 3. Meanwhile, for the variation of parameters and numerical approaches, the results are not comparable, and result relatively have large errors as seem in Figure 4. However, the comparison between LSM and CG Method has shown identical results (Refer Figure 4).

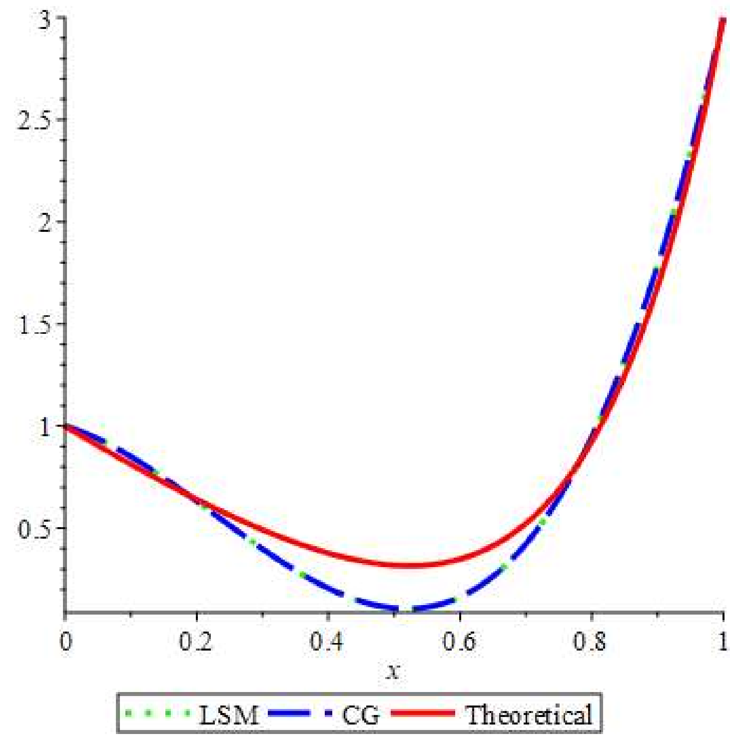

Figure 2. Graph of results for $y'' - 3y' - 4y = 3e^{2x}$, BVP:$y(0) = 1, y(1) = 3$

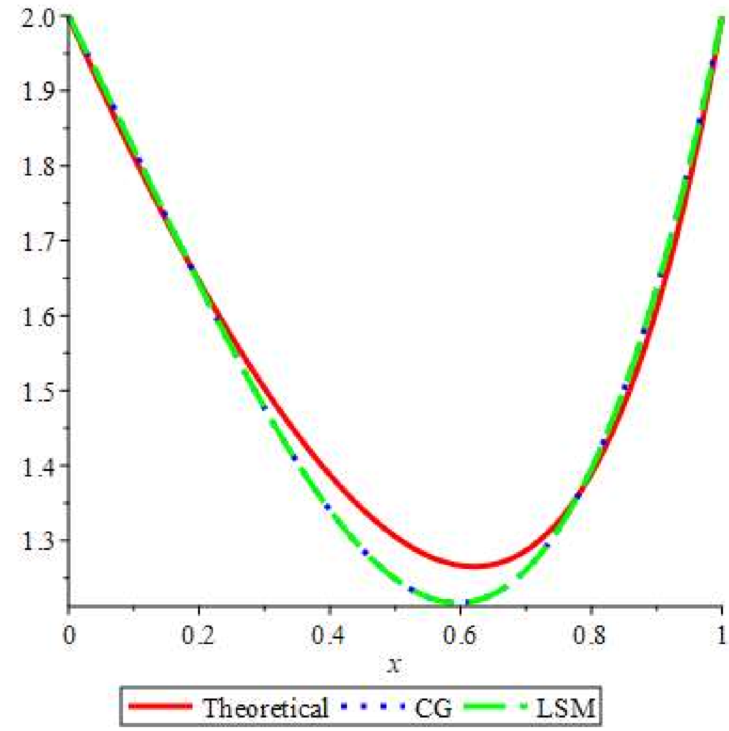

Figure 3. Graph of results for $y'' - 3y' - 4y = 2\sin x$, BVP:$y(0) = 2, y(1) = 2$

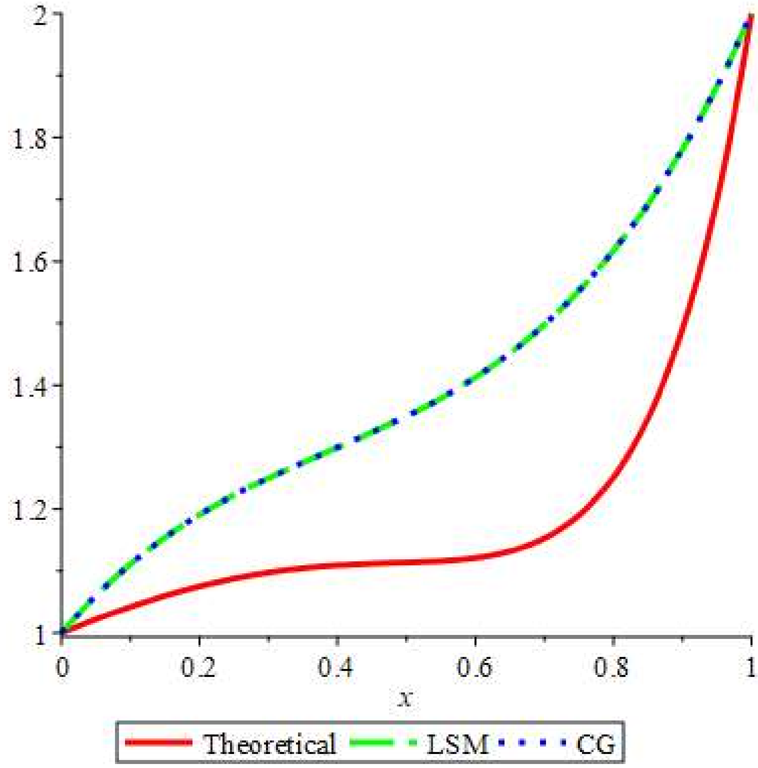

Figure 4. Graph of results for $y'' - 5y' + 4y = e^{3x}$, BVP:$y(0) = 1, y(1) = 2$

5. Conclusion

Three problems of second order nonhomogeneous linear ODE have been selected in this research. Only one problem was resolved using the VP approach, while two problems were calculated using the UC method. This study shows that LSM and CG can be used as an alternative to solve second order nonhomogeneous linear ODE. Even if there were some inaccuracies in the results, the graph still resembles one to another. LSM can be adapted to boundary value problem given in terms of the function. Moreover, it is also a practical method that is easy to implement. Based on the results analysis, LSM is a good way of solving second order nonhomogeneous linear ODE. Meanwhile, the CG method is to overcome the problem of singular and non-singular matrix as in the final part of LSM.

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Conflict of Interest
The authors declare no conflict of interest in the subject matter or materials discussed in this manuscript.

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